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TRANSIENT RADIATION AND CONDUCTION IN AN ABSORBING, EMITTING, SCATTERING SLAB WITH REFLECTIVE BOUNDARIES

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THE STEADY-STATE heat transfer by simultaneous radiation and conduction in an absorbing and emitting plane-parallel medium has been studied by several investigators [1–6] and the scattering effect included in the analysis [7, 8], but the non-steady case has been studied for an absorbing and emitting medium [9–11]. In the present investigation, transient conduction and radiation in an absorbing, emitting, isotropically scattering slab with reflective boundaries is solved by the application of Case's normal-mode expansion technique to the radiation part of the problem. This technique, developed by Case [12] for solving one-dimensional neutron transport problems, has been applied recently in the solution of pure radiative heat transfer problems [13–15]. The purpose of this paper is to demonstrate the application of the normal-mode expansion technique to the solution of unsteady conduction with radiation and to present highly accurate results against which the accuracy of various approximate solutions can be checked.

ANALYSIS

Consideration is given to the problem of simultaneous radiation and conduction in an absorbing, emitting, isotropically scattering, gray, plane-parallel, constant property slab of optical thickness τ_0, initially at zero temperature, and for times greater than zero the two boundary surfaces at $\tau = 0$ and $\tau = \tau_0$ are maintained at uniform temperatures T_1 and T_2 respectively. The one-dimensional time dependent energy equation for combined conduction and radiation is given in the dimensionless form as:

$$\frac{\partial^2}{\partial \tau^2}\theta(\tau,\xi) - \frac{1}{4\pi N}\frac{\partial}{\partial \tau}Q^r(\tau,\xi) = \frac{\partial}{\partial \xi}\theta(\tau,\xi), \quad \text{in } 0 < \tau < \tau_0 \tag{1}$$

with the boundary and initial conditions

$$\theta(0,\xi) = 1, \theta(\tau_0,\xi) = \theta_2 \quad \text{and} \quad \theta(\tau,0) = 0, \tag{2}$$

where we have defined the following dimensionless quantities

$$N = \frac{k\beta}{4n^2\sigma T_1^3} = \text{conduction-to-radiation parameter},$$

$$Q^r(\tau,\xi) = \frac{q^r(\tau,\xi)}{n^2\sigma T_1^4/\pi} = \text{dimensionless net radiative heat flux},$$

$$\theta(\tau,\xi) = \frac{T(\tau,\xi)}{T_1} = \text{dimensionless temperature},$$

$$\xi = \frac{k\beta^2 t}{\rho C_p} = \text{dimensionless time}.$$

Here C_p is the specific heat, k is the thermal conductivity, n is the refractive index, β is the extinction coefficient, σ is the Stefan–Boltzmann constant and ρ is the density.

The energy equation includes the net radiative heat flux $Q^r(\tau,\xi)$ which is related to the dimensionless intensity $\psi(\tau,\mu,\xi)$ by

$$Q^r(\tau,\xi) = 2\pi \int_{-1}^{1} \psi(\tau,\mu,\xi)\,\mu\,d\mu \tag{3}$$

where $\psi(\tau,\mu,\xi)$ satisfies the equation of radiation transfer

$$\mu\frac{\partial}{\partial \tau}\psi(\tau,\mu,\xi) + \psi(\tau,\mu,\xi) = (1-\omega)\,\theta^4(\tau,\xi) + \frac{\omega}{2}\int_{-1}^{1}\psi(\tau,\mu',\xi)\,d\mu', \quad -1 \leqslant \mu \leqslant 1 \tag{4}$$

with the boundary conditions

$$\psi(0,\mu,\xi) = \varepsilon_1 + \rho_1^s\psi(0,-\mu,\xi) + 2\rho_1^d\int_0^1 \psi(0,\mu',\xi)\,\mu'\,d\mu', \quad \mu > 0 \tag{5}$$

$$\psi(\tau_0,-\mu,\xi) = \varepsilon_2\theta_2^4 + \rho_2^s\psi(\tau_0,\mu,\xi) + 2\rho_2^d\int_0^1 \psi(\tau_0,\mu',\xi)\,\mu'\,d\mu', \quad \mu > 0. \tag{6}$$

Here ω is the single scattering albedo, μ is the cosine of the angle between the direction of the radiation intensity and the positive τ axis. The boundary surfaces are considered to have reflectivities ρ_i which can be expressed as a sum of a specular ρ_i^s and diffuse ρ_i^d reflectivity components in the form

$$\rho_i = \rho_i^s + \rho_i^d, \quad i = 1,2 \tag{7}$$

where 1 and 2 refer to the boundaries at $\tau = 0$ and $\tau = \tau_0$ respectively.

The solution of equation (4) can be written in the form [14]

$$\psi(\tau,\mu,\xi) = A(\eta_0,\xi)\,\phi(\eta_0,\mu)\,e^{-\tau/\eta_0} + A(-\eta_0,\xi)\,\phi(-\eta_0,\mu)\,e^{\tau/\eta_0} + \int_0^1 A(\eta,\xi)\,\phi(\eta,\mu)\,e^{-\tau/\eta}\,d\eta + \int_0^1 A(-\eta,\xi)\,\phi(-\eta,\mu)\,e^{\tau/\eta}\,d\eta + \psi_p(\tau,\mu,\xi) \tag{8}$$

where $A(\pm\eta_0,\xi)$ and $A(\pm\eta,\xi)$, $\eta\in(0,1)$ are the expansion coefficients, $\psi_p(\tau,\mu,\xi)$ is a particular solution of equation (4), and $\phi(\pm\eta_0,\mu)$ and $\phi(\pm\eta,\mu)$ are the normal modes [16].

From equations (3) and (8) we obtain

$$Q^r(\tau,\xi) = 2\pi(1-\omega)\Big[A(\eta_0,\xi)\,\eta_0\,e^{-\tau/\eta_0} - A(-\eta_0,\xi)\eta_0\,e^{\tau/\eta_0} + \int_0^1 A(\eta,\xi)\,\eta\,e^{-\tau/\eta}\,d\eta - \int_0^1 A(-\eta,\xi)\,\eta\,e^{\tau/\eta}\,d\eta + \frac{1}{1-\omega}\int_{-1}^{1}\psi_p(\tau,\mu,\xi)\,\mu\,d\mu\Big]. \tag{9}$$

Then $\partial Q^r(\tau,\xi)/\partial\tau$ can be obtained by differentiating equation (9) with respect to τ. The expansion coefficients $A(\pm\eta_0,\xi)$ and $A(\pm\eta,\xi)$ can be determined by constraining the solution equation (8) to meet the boundary conditions equations (5) and (6), and by utilizing the orthogonality property of the normal modes and various normalization integrals as described in references [14, 15] provided that a particular solution $\psi_p(\tau,\mu,\xi)$ is available. To determine a particular solution it is assumed that an initial guess is available for the temperature distribution $\theta^0(\tau,\xi)$ and the inhomogeneous term is represented as a polynomial in τ in the form
a

$$(1-\omega)[\theta^0(\tau,\xi)]^4 = (1-\omega)\sum_{n=0}^{N} B_n(\xi)\,\tau^n \tag{10}$$

and that the coefficients $B_n(\xi)$ are determined. Noting that a particular solution $\psi_{p,n}(\tau,\mu)$ for an inhomogeneous term of the form $(1-\omega)\,\tau^n$ is given as [17]

$$\psi_{p,n}(\tau,\mu) = \sum_{r=0,1,2,\ldots}^{n} (-1)^r \frac{n!}{(n-r)!}\tau^{n-r}\mu^r + \frac{\omega}{1-\omega}\sum_{s=1,2,3..} \frac{n!}{(n-2s)!\,(2s+1)}\psi_{p,n-2s}(\tau,\mu) \tag{11}$$

$\frac{n-1}{2}$ (n odd)

$\frac{n}{2}$ (n even)

then, the particular solution $\psi_p(\tau,\mu,\xi)$ for $(1-\omega)[\theta^0(\tau,\xi)]^4$ is given by

$$\psi_p(\tau,\mu,\xi) = \sum_{n=0}^{N} B_n(\xi)\,\psi_{p,n}(\tau,\mu). \tag{12}$$

By treating $(\partial/\partial\tau)\,Q^r(\tau,\xi)$ as a prescribed function, the heat conduction problem given by equations (1) and (2) is integrated by the application of a finite integral transform technique [18] to yield

$$\theta(\tau,\xi) = 1 + (\theta_2 - 1)\frac{\tau}{\tau_0} - \frac{2}{\tau_0}\sum_{m=1}^{\infty}\frac{1-(-1)^m\theta_2}{v_m}e^{-v_m^2\xi}\sin v_m\tau - \frac{2}{\tau_0}\sum_{m=1}^{\infty} e^{-v_m^2\xi}\sin v_m\tau\,\frac{1}{4\pi N}\int_{\xi'=0}^{\xi} e^{v_m^2\xi'}\int_{\tau'=0}^{\tau_0}\sin v_m\tau'\,\frac{\partial}{\partial\tau'}Q^r(\tau',\xi')\,d\tau'\,d\xi' \tag{13}$$

where $v_m = m\pi/\tau_0$, m = 1, 2, 3... The actual temperature distribution $\theta(\tau,\xi)$ in the medium can be evaluated from equation (13) by an iterative scheme.

For the steady-state the equation to be iterated becomes

$$\theta(\tau) = 1 + (\theta_2 - 1)\frac{\tau}{\tau_0} + \frac{1}{4\pi N}\left[\int_0^{\tau} Q^r(\tau')\,d\tau' - \frac{\tau}{\tau_0}\int_0^{\tau} Q^r(\tau')\,d\tau'\right] \tag{14}$$

where $Q^r(\tau)$ is given by equation (9) except it does not involve the time variable ξ. Once the temperature distribution in the medium is known, the heat transfer quantities are readily evaluated. For example, the total heat flux in the medium is equal to the sum of the conductive and radiative heat fluxes, and in the dimensionless form it is given by

$$Q(\tau,\xi) \equiv \frac{q(\tau,\xi)}{k\beta T_1} = -\frac{\partial}{\partial\tau}\theta(\tau,\xi) + \frac{1}{4\pi N}Q^r(\tau,\xi). \tag{15}$$

NUMERICAL CALCULATIONS AND RESULTS

All computations are performed in double-precision on the IBM 360/75 computer. The expansion coefficients $A(\pm\eta_0,\xi)$ and $A(\pm\eta,\xi)$ are solved as described in reference [15] by an iterative process with the integrands being evaluated by a 41-point improved Gaussian quadrature scheme [19] and the results are converged to 12 significant digits. Several checks have been employed to substantiate our confidence in the accuracy of the results reported here.

Table 1. Effect of parameters N *and* ω *on steady-state conductive, radiative and total heat fluxes*
$\theta_1 = 1, \quad \theta_2 = 0, \quad \tau_0 = 1, \quad \varepsilon_1 = \varepsilon_2 = 1$

N	ω	$-\frac{\partial\theta}{\partial\tau}$, $\frac{\tau}{\tau_0} = 0$	0·5	1·0	$\frac{Q^r}{4\pi N}$, $\frac{\tau}{\tau_0} = 0$	0·5	1·0	Total heat flux q: $\frac{q}{k\beta T_1} = -\frac{\partial\theta}{\partial\tau} + \frac{Q^r}{4\pi N}$
0·5	0	0·9396	0·9879	1·1447	0·3585	0·3102	0·1534	1·2981
	0·5	0·9491	0·9930	1·0983	0·3392	0·2954	0·1900	1·2884
	0·9	0·9798	0·9985	1·0305	0·2995	0·2803	0·2488	1·2793
	1·0	1·0	1·0	1·0	0·2767	0·2767	0·2767	1·2767
0·1	0	0·8520	0·8799	1·6707	1·6651	1·6372	0·8464	2·5171
	0·5	0·8513	0·9386	1·4366	1·5924	1·5051	1·0071	2·4437
	0·9	0·9084	0·9901	1·1486	1·4896	1·4080	1·2494	2·3980
	1·0	1·0	1·0	1·0	1·3835	1·3835	1·3835	2·3835
0·05	0	0·8986	0·7159	2·2187	3·1544	3·3371	1·8344	4·0530
	0·5	0·7908	0·8314	1·8689	3·0937	3·0531	2·0156	3·8845
	0·9	0·8382	0·9745	1·2877	2·9608	2·8246	2·5114	3·7991
	1·0	1·0	1·0	1·0	2·7670	2·7670	2·7670	3·7670

The accuracy of the expansion coefficients $A(\pm\eta_0, \xi)$ and $A(\eta, \xi)$ is tested by checking the moments of the boundary conditions for the radiation part of the problem. The agreement for various moments has been found within five or six significant digits. The coefficients $B_n(\xi)$ are determined for trial cases by using both a Gaussian elimination and a linear least squares. The Gaussian elimination is preferred because it provides exact representation at the desired locations, especially at the boundaries. Since we have performed the calculations with a high degree of accuracy and employed several checks, our results may serve as "bench mark" solutions to check the accuracy of various approximate methods.

We present in Table 1 the results for the steady-state conductive, radiative and the total heat fluxes for several different values of the parameters N, ω for a set of conditions shown on this table.

We present in Tables 2 and 3 the effects of single scattering albedo ω and the wall reflectivities on transient temperature distribution, the conductive, the radiative and the total heat

Table 2. Effects of ω *and wall reflectivities on transient temperature distribution at dimensionless time* $\xi = 0·05$ *for* $\theta_1 = 1$, $\theta_2 = 0$, $\tau_0 = 1·0$ *and* N $= 0·1$

ω	ε_1	ρ_1^s	ρ_1^d	ε_2	ρ_2^s	ρ_2^d	$\theta(\tau,\xi)$, $\frac{\tau}{\tau_0} = \frac{1}{4}$	$\frac{\tau}{\tau_0} = \frac{2}{4}$	$\frac{\tau}{\tau_0} = \frac{3}{4}$
0	1	0	0	1	0	0	0·4991	0·1872	0·0644
	1	0	0	0	1	0	0·5089	0·2036	0·0827
	1	0	0	0	0	1	0·5085	0·2039	0·0844
	0·5	0·5	0	0·5	0·5	0	0·4699	0·1627	0·0517
	0·5	0	0·5	0·5	0	0·5	0·4694	0·1628	0·0525
0·5	1	0	0	1	0	0	0·4617	0·1474	0·0277
	1	0	0	0	1	0	0·4748	0·1673	0·0581
	1	0	0	0	0	1	0·4716	0·1630	0·0545
	0·5	0·5	0	0·5	0·5	0	0·4384	0·1264	0·0250
	0·5	0	0·5	0·5	0	0·5	0·4323	0·1196	0·0195
1·0	(or pure conduction)						0·4292	0·1138	0·0176

Table 3. Effects of ω and wall reflectivities on transient conductive and radiative heat fluxes at dimensionless time $\xi = 0{\cdot}05$ for $\theta_1 = 1, \theta_2 = 0, \tau_0 = 1{\cdot}0$ and $N = 0{\cdot}1$

ω	ε_1	ρ_1^s	ρ_1^d	ε_2	ρ_2^s	ρ_2^d	$-\frac{\partial}{\partial\tau}\theta(\tau,\xi)$			$Q^r(\tau,\xi)/4\pi N$		
							$\frac{\tau}{\tau_0}=0$	$\frac{\tau}{\tau_0}=\frac{2}{4}$	$\frac{\tau}{\tau_0}=1$	$\frac{\tau}{\tau_0}=0$	$\frac{\tau}{\tau_0}=\frac{2}{4}$	$\frac{\tau}{\tau_0}=1$
0	1	0	0	1	0	0	2·2974	0·7972	0·2980	2·1260	1·2923	0·6303
	1	0	0	0	1	0	2·2428	0·7763	0·4521	1·9480	0·9732	0
	1	0	0	0	0	1	2·2470	0·7714	0·4833	1·9812	1·0172	0
	0·5	0·5	0	0·5	0·5	0	2·5562	0·7542	0·2451	1·0482	0·7041	0·1927
	0·5	0	0·5	0·5	0	0·5	2·5677	0·7513	0·2566	1·0529	0·7248	0·1956
0·5	1	0	0	1	0	0	2·4396	0·7625	0·1326	1·6436	1·2529	0·9746
	1	0	0	0	1	0	2·3644	0·7439	0·2988	1·3594	0·7857	0·1663
	1	0	0	0	0	1	2·3835	0·7450	0·2765	1·4000	0·9084	0·3411
	0·5	0·5	0	0·5	0·5	0	2·6349	0·7345	0·0689	0·6015	0·5978	0·4856
	0·5	0	0·5	0·5	0	0·5	2·6802	0·7330	0·0313	0·5317	0·6188	0·5884
1	1	0	0	1	0	0	2·5231	0·7229	0·0340	1·3835	1·3835	1·3835
	1	0	0	0	1	0	2·5231	0·7229	0·0340	0	0	0
	0·5	0·5	0	0·5	0·5	0	2·5231	0·7229	0·0340	0·6523	0·6523	0·6523
	0·5	0	0·5	0·5	0	0·5	2·5231	0·7229	0·0340	0·8906	0·8906	0·8906

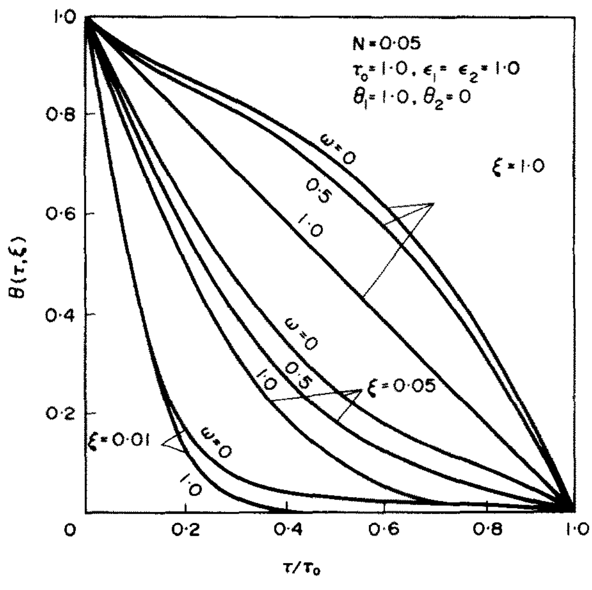

FIG. 1. Effects of radiation on transient temperature distribution.

fluxes at a dimensionless time $\xi = 0{\cdot}05$ for $\theta_1 = 1$, $\theta_2 = 0$, $\tau_0 = 1$ and $N = 0{\cdot}1$. Figure 1 shows the effects radiation on temperature profile during transients at three different dimensionless times $\xi = 0{\cdot}01$, $0{\cdot}05$ and $1{\cdot}0$ for $\omega = 0$, $0{\cdot}5$ and $1{\cdot}0$. For the problem considered here the case $\xi = 1{\cdot}0$ is almost identical to the steady state temperature profile. The curves for $\omega = 1$ characterizes the temperature profile for pure conduction since radiation does not interact with conduction in a purely scattering medium. The data in Table 2 shows that there is only slight difference in temperature with specularly and diffusely reflecting cases. Radiation appears to have negligible effect on temperature distribution at early times, but it has significant effect on temperature distribution at later times as shown in Fig. 1.

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TRANSIENT MASS TRANSPORT BETWEEN A FINITE VOLUME OF HOMOGENIZED FLUID AND A SPHERE WITH FINITE INTERFACIAL TRANSPORT COEFFICIENTS

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NOMENCLATURE

b, root of equations (7) or (9);
C, solute concentration (or temperature) in sphere;
D, diffusion coefficient (or thermal diffusivity)
F, solute concentration (or temperature) in fluid;
k, mass transfer coefficient (or heat transfer coefficient/volumetric heat capacity of sphere);
r, radial distance from sphere center;